Mersenne Primes and perfect numbers - Solution

1. Prime numbers.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197...

2. Perfect numbers.

Verify for yourself that the numbers 28 and 496 are in fact perfect numbers, by completing the table below. You may use a calculator to work out the answers.

6	$=1+2+3$
28	$=1+2+4+7+14$
496	$=1+2+4+8+16+31+62+124+248$

3. How to find perfect numbers?

Try to predict:

1. the number of digits in the 5th perfect number,
   5 digits
2. the last digit of the 5th perfect number.
   the last digit would be 6

	Sum	Prime	Calculation	Perfect number
$1+2$	$=3$	✓	2×3	6
$1+2+4$	$=7$	✓	4×7	28
$1+2+4+8$	$=15$	✗		
$1+2+4+8+16$	$=31$	✓	16×31	496
$1+2+4+8+16+32$	= 63	x		
1 +....+ 64	=127	✓	127×64	8128
1 +....+ 128	=255	x		
1 +....+ 256	=511	X(7x73)		
1 +....+ 512	=1023	x		
1 +....+ 1024	=2047	x		
1 +....+ 2048 <- you won, do you want to continue?	=4095	x		
1 +....+ 4096	=8191	✓	8191×4096	33550336
1 +....+ 8192	=13383	x		

(a) Express the first two columns of numbers from the previous table in terms of powers of two.

Series	**Sum**
$1+2^1$	2^2-1
$1+2^1+2^2$	2^3-1
$1+2^1+2^2+2^3$	2^4-1
$1+2^1+2^2+2^3+2^4$	2^5-1
$1+2^1+2^2+2^3+2^4+2^5$	2^6-1
$1+2^1+2^2+2^3+2^4+2^5+2^6$	2^7-1

(b) Using the shortcut calculation you have just learned, compute the following sum:

$$1+2^1+2^2+2^3+2^4+2^5+2^6+.....+2^{24}+2^{25}=2^{26}-1=67108863$$

(c) Generalization: write down the formula for the sum of the first n terms of the series:

$$1+2^1+2^2+2^3+2^4+2^5+.....+2^{n-1}=2^n-1$$

(d) What is the largest power of two whose digits can be displayed fully on your calculator? 2^{33} ? 2^99?

4. Mersenne primes.

(a) Complete the following table, expressing the first five Mersenne primes and perfect numbers in powers of two.

Mersenne prime		**Perfect number**	
3	2^2-1	$2^1\left(2^2-1\right)$	6
7	2^3-1	$2^2\left(2^3-1\right)$	28
31	2^5-1	$2^4\left(2^5-1\right)$	496
127	2^7-1	$2^6\left(2^7-1\right)$	8128
8191	$2^{13}-1$	$2^{12}\left(2^{13}-1\right)$	33.550.336
	$2^{17}-1$	$2^{16}\left(2^{17}-1\right)$	8.589.869.056

(b) Two perfect numbers were discovered in 1588, both by Cataldi. These two perfect numbers can be obtained from the Mersenne primes $M_{17} = 2^{17} - 1$ and $M_{19} = 2^{19} - 1$. Can you compute these two perfect numbers with the help of your calculator?
M_{17} = 131.071; Perfect number 8.589.869.056
M_{19} = 524.287; Perfect number 1.374386913×10^{11}, I can't compute it without calculator!

(c) Do you think M_{21} is a Mersenne prime?
I don't think so, because 21 is not prime. Moreover, M_{21}=2.097.151=7×299.593

Mn was known for : n = **2**, **3**, **5**, **7**, **13**, **17**,**19**, 31, 67, 127 and 257 (*)

5. How to find Mersenne primes?

Study carefully the list of Mersenne primes which have been established so far:
$M_2,\ M_3,\ M_5,\ M_7,\ M_{13},\ M_{17},\ M_{19},\ \ldots$

Do the values of *n* in these numbers provide any clue to the pattern of occurrence of Mersenne primes?
M_p seems to be prime only if p is prime; but M_{11} is missing from the list.

(a) How many Mersenne primes were known during Mersenne's time?
11 of them were known (see list (*) above).

(b) Without using calculator, show that $M_{14} = 2^{14} - 1$ can be factorized. Do the same for $M_{26} = 2^{26} - 1$.
[*Hint*: Express M_{14} as a difference of two squares.]

$$M_{14}=2^{14}-1=\left(2^7\right)^2-1^2=\left(2^7-1\right)\left(2^7+1\right)$$

$$M_{26}=2^{26}-1=\left(2^{13}\right)^2-1^2=\left(2^{13}-1\right)\left(2^{13}+1\right)$$

As they can be factorized, they aren't prime (they are composite).

(c) Prove that, if *n* is an <u>even</u> number greater than 2, then $2^n - 1$ is composite.
Let n be even, there is an integer p such that n=2p.
$2^n-1=2^{2p}-1=\left(2^p\right)^2-1^2=\left(2^p-1\right)\left(2^p+1\right)$, and 2^n-1 is composite.

(d) Without using calculator, show that $M_{15} = 2^{15} - 1$ can be factorized. Do the same for $M_{21} = 2^{21} - 1$.

[*Hint*: $1 + r + r^2 + \ldots + r^{n-1} = \dfrac{r^n - 1}{r - 1}$]

$$\frac{2^{15}-1}{2-1} = 1+2+2^2+....+2^{14}$$
$$\frac{2^{15}-1}{1} = 1+2+2^2+....+2^{14}$$
$$2^{15}-1 = 1+2+2^2+....+2^{14}$$
$$2^{15}\cancel{-1}\cancel{+1} = 1+1+2+2^2+....+2^{14}$$
$$2^{15} = 2+2+2^2+....+2^{14}$$
$$2^{15} = 2(1+1+2+....+2^{13})$$

$$\frac{2^{21}-1}{2-1} = 1+2+2^2+....+2^{20}$$
$$\frac{2^{21}-1}{1} = 1+2+2^2+....+2^{20}$$
$$2^{21}-1 = 1+2+2^2+....+2^{20}$$
$$2^{21}\cancel{-1}\cancel{+1} = 1+1+2+2^2+....+2^{20}$$
$$2^{21} = 2+2+2^2+....+2^{20}$$
$$2^{21} = 2(1+1+2+....+2^{19})$$

(e) Check that, for any integers a and b, $\left(2^a - 1\right)\left(1 + 2^a + 2^{2a} + \ldots + 2^{(b-1)a}\right) = 2^{ab} - 1$.

$$\left(2^a - 1\right)\left(1 + 2^a + 2^{2a} + \ldots + 2^{(b-1)a}\right)$$
$$= \cancel{2^a} \cancel{+2^{2a}} + 2^{3a} + \ldots \cancel{+2^{(b-1)a}} + 2^{ba} - 1 \cancel{-2^a} \cancel{-2^{2a}} - \ldots \cancel{-2^{(b-1)a}}$$
$$= 2^{ba} - 1$$

(f) Prove that, if n is composite, then $2^n - 1$ is composite.

If n is composite, it means that n can be written as a multiplication between two integers. Let a and b be those integers, we get $n = ab$.